teorema
Vol. XXXIII/2, 2014, pp. 151-170
ISSN: 0210-1602
[BIBLID 0210-1602 (2014) 33:2; pp. 151-170]

Truth and Probability in Game-Theoretical Semantics

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RESUMEN

En este artículo describiremos brevemente el sistema que en lógica es conocido como lógica IF (lógica amigable con la independencia) y que fue introducido por Hintikka y Sandu en 1989. Es conocido que esta lógica tiene enunciados que son indeterminados. Tras esto, mostraremos cómo resolver la indeterminación de sus enunciados aplicando el teorema Minimax de von Neumann. Este artículo se basa en gran medida en [Sevenster y Sandu (2010)], [Mann, Sandu, y Sevenster (2011)], [Sandu (2012)], [Sandu (en prensa)], and [Barbero and Sandu (en prensa)].

PALABRAS CLAVE: *semántica de teoría de juegos, información imperfecta, equilibrio de Nash, teorema minimax*.

ABSTRACT

In this paper we shall shortly describe the system of logic known as IF logic (Independence friendly logic) introduced in Hintikka and Sandu (1989). It is known that this logic has indeterminate sentences. After that we will show how we can resolve the indeterminacy of its sentences by applying von Neumann's Minimax theorem. This paper draws heavily on [Sevenster and Sandu (2010)], [Mann, Sandu, and Sevenster (2011)], [Sandu (2012)], [Sandu (forthcoming)], and [Barbero and Sandu (forthcoming)].

KEYWORDS: *Game-Theoretical Semantics, Imperfect Information, Nash Equilibrium, Minimax Theorem*.

I. SHORT HISTORICAL BACKGROUND

Goldfarb (1979) shows how sinuous the development of logic in the XXth. century was. Modern quantification theory arose at the crosswords of different conceptions of the nature of the quantifier:

- The algebraic view: Peirce, Schröder, and Löwenheim
- Quantifiers as second-order properties: Frege, Montague
- Quantifier-dependence and choice functions: Skolem and Hilbert.

The algebraic school assimilates quantifiers to (possibly infinite) sums and products whose contributions to the relevant formal system are given through algebraic manipulations. Even if Löwenheim made a distinction between quantification over individuals and quantification over relations which opened the door to the separation of first-order from second-order logic, his interest in the first-order fragment of the calculus seems motivated by purely algebraic, rather than foundational considerations [Goldfarb (1979)]. By viewing quantifiers as (higher-order) relations over the universe, Frege introduced content into logic and broke with the algebraic tradition.

Skolem and Hilbert broke both with the algebraic tradition and with Frege's conception of quantifiers as higher-order relations. Both of them recognized something special about the nature of quantification: the phenomenon of quantifier dependence, that is, the idea of a quantifier depending on others. Both expressed quantifier-dependence through the use of choice functions. Skolem used quantifier-dependence to give an alternative proof of Löwenheim's theorem. Hilbert, unlike Skolem, was concerned with the role of quantifiers informal proofs, and his use of the choice functions encoded in the ε-terms is subordinated entirely to this purpose:

> I am suggesting that behind Hilbert's interest in proving by finitistic hook or crook, the consistency of formal systems, lies a deeper point: that of using the proxy choice-functions to provide in some measure an explication of the meaning of the quantification used in formal proofs [Goldfarb (1979), p.361].

Given that any formal proof is finite, it contains only a finite number of ε-axioms. The idea is now to assign successively, during the proof, effective values to ε -terms with the hope of transforming the whole proof in a manipulation of quantifier-free formulas. To this purpose Hilbert did not need the full power of the choice functions; it sufficed, instead "to obtain finitely-based functions (functions that are zero everywhere but on a finite number of arguments) that approximate the "real" choice functions" [Goldfarb (1979), p. 361].

All in all, Goldfarb shows convincingly how the connection between quantifier dependence and choice functions, is at the heart of how classical logicians in the twenties viewed the nature of quantification.

II. Dependence and Independence of Quantifiers

Hintikka's game-theoretical semantics (GTS) with its main interest on the phenomenon of quantifier dependence and independence for the foundations of mathematics continues the Skolem-Hilbert tradition. The connection between the satisfiability of a sentence and the existence of strategies (Skolem functions) is a natural byproduct of this analysis. In this paper I will focus on the extension of the patterns of dependence and independence of quantifiers beyond those which may be expressed in ordinary first-order logic. I will start with few examples (inspired by [Tao (2007)]).

If $B(x,y)$ is a binary relation, then we can express in ordinary first-order logic the statement

1. For every x, there exists a y depending on x such that $B(x, y)$ is true

by

$\forall x \exists y B(x, y)$.

Similarly, we can express the statement

2. For every x, there exists a y independent of x such that $B(x, y)$ is true

by

$\forall y \exists x B(x, y)$.

If $C(x, y, z)$ is a ternary relation, then we can express the statement

3. For every x, and y, there exists a z independent from x and y such that $C(x, y, z)$ is true

by

$\exists z \forall x \forall y \mathrm{C}(x, y, z)$.

An example from mathematics which exemplifies such a pattern is the definition of a Lipchitz-continuous function. A function f: $R \rightarrow R$ is said to be Lipchitz-continuous if

$\exists z \forall x \forall y\ (|f(x) - f(y)| \leq z \cdot |x - y|)$.

When $D(x, y, z, w)$ is a quaternary relation, we can express the statement

4. For every x, y, and z there exists a w depending on x and y but independent from z such that $D(x, y, z, w)$ is true

by

$\forall x \forall y \exists w \forall z\ D(x, y, z, w)$.

A typical example of this quantifier dependence and independence in mathematics is the *delta-epsilon* definition of the *continuity* of a function. A function f is said to be continuous at a point x_0 if given any $\varepsilon > 0$ one can choose $\delta > 0$ so that for all y, when x_0 is within distance δ from y, then $f(x_0)$ is within distance ε from $f(y)$, i.e.,

$$|x_0 - y| < \delta \rightarrow |f(x_0) - f(y)| < \varepsilon.$$

The general form of this definition is

$$\forall x_0 \forall \varepsilon \exists \delta \forall y D(x_0, \varepsilon, \delta, y).$$

Notice that the choice of δ depends on both x_0 and ε.
One can similarly express the statement

5. For every x, y, and z there exists a w depending on y but independent from x and z such that $D(x, y, z, w)$ is true

by rearranging the quantifiers in (4) to obtain

$$\forall y \exists w \forall x \forall z D(x, y, z, w).$$

The difference between (4) and (5) is exemplified by the difference between a function being continuous and it being *uniformly continuous*: f is said to be uniformly continuous if in the general definition above the choice of δ depends only on ε (and not on the point x_0). That is, uniform continuity is expressed by

$$\forall \varepsilon \exists \delta \forall x_0 \forall y D(x_0, \varepsilon, \delta, y).$$

The dependencies and independencies within a group of four quantifiers, two universal and two existentials becomes more complex. We can still express in first-order logic the statement

6. For every x and z, there exists a y depending on x and z and a w depending only on z such that $D(x, z, y, w)$ is true

by

$$\forall z \exists w \forall x \exists y D(x, z, y, w)$$

but we cannot always express the statement

7. For every x and z, there exists a y depending only on x and a w depending only on z such that $D(x, z, y, w)$ is true

neither the statement

8. There exists a w such that for every x, there exists a y which depends only on x, and a z which depends only on y such that $D(x, z, y, w)$ is true.

Examples will be given in the next section.

III. INDEPENDENCE-FRIENDLY LOGIC

IF logic (Independence-Friendly logic) is an extension of first-order logic which contains quantifiers and connectives of the form

$(\exists x/W)$, $(\forall x/W)$, $(\vee/W)$, $(\wedge /W)$

where the interpretation of e.g. $(\exists x/W)$ is: "the choice of x is independent of the values of the variables in W". When $W = \emptyset$, we recover the standard quantifiers and connectives. One can express in this logic patterns of dependence and independence of quantifiers which are not expressible in ordinary first-order logic. For instance (7) above can be expressed by

$\forall x \forall z(\exists y/\{z\})(\exists w/\{x, y\})D(x, z, y, w)$

and (8) by

$\exists w \forall x(\exists y/\{w\})(\exists z/\{w, x\})D(w, x, y, z)$.

There are three equivalents semantic interpretations for IF formulas:

- Semantical games of imperfect information [Hintikka and Sandu (1989); (1997)]
- Compositional semantics (trump semantics, [Hodges(1997)])
- Skolem functions and Kreisel counter-examples [Sevenster and Sandu (2010)]

The equivalence of the three interpretations is shown in [Mann, Sandu, and Sevenster (2011)].

III.1 *Semantical Games of Imperfect Information*

Very briefly, a semantical (extensive) game of imperfect information, $G(\mathbb{M},s,\varphi)$, is associated with an IF formula (in negation normal form), a model $\mathbb{M}$ and an assignment s which includes the free variables of φ. The game is played by two players, $\exists$ (Eloise) and $\forall$(Abelard). Eloise's moves are prompted by existential quantifiers ($(\exists x/W)$) and disjunctions: she chooses an individual from the domain to be the value of the existentially quantified variable; for a disjunction, she chooses one of the disjuncts. The moves of Abelard are the duals of those of Eloise. Any play of the game stops with an "atomic position" (A, r), where A is an atomic formula or its negation and r is an assignment which includes the free variables of A. If r satisfies A in $\mathbb{M}$, then $\exists$ wins the play. Otherwise $\forall$ wins it.

A *strategy* for player p, is, intuitively, a (deterministic) method σ which gives p a choice for every position where p is to move. Leaving things at an informal level, σ is required to be *uniform*, which, making a long story short, amounts to its arguments being only those positions in a play which p "sees". σ is a winning strategy if p wins every possible play where she follows σ.

Truth and *falsity* (with respect to an assignment) are then defined by:

- $\mathbb{M}, s \vDash^{+}_{GTS} \Psi$ iff there is a winning strategy for $\exists$ in G($\mathbb{M}$, s, Ψ)
- $\mathbb{M}, s \vDash^{-}_{GTS} \Psi$ iff there is a winning strategy for $\forall$ in G($\mathbb{M}$, s, Ψ).

When ψ is an ordinary (slash-free) first order formula, one can use Zermelo's Theorem to show that every game G($\mathbb{M}$, s, Ψ) is determinate: either Eloise has a winning strategy or Abelard has a winning strategy. For ψ an IF formula, this theorem may fail, as we will see below. Let us give a couple of examples which illustrate the notion of strategy in the context of imperfect information. Both are taken from Janssen and Duchesne (2006).

We show that the IF sentence

$$\forall x \forall z(x \neq z \vee (\exists y/\{x\})x = y)$$

is a logical truth (true in every model with at least two elements). So let M be a set with at least two elements. Here is a winning strategy for Eloise:

- Let $\forall$ choose $x, z \in M$. If $x \neq z$, then $\exists$ chooses Left. If $x = z$, then $\exists$ chooses Right after which she chooses z (that she "sees".)

Next we show that the IF sentence

$$\exists x \exists y \exists z[x = y \wedge \forall v(\forall u/\{x\})(u \neq x \vee v \neq z)]$$

is a logical falsity. Let M be a set with two elements. Here is a winning strategy for Abelard:

• Let ∃ choose x, y, z. If $x \neq y$ then chooses Left. If $x = y$ then ∀ chooses Right, and then he chooses $v = z$ (that she sees) and $u = y$ that he also sees. Given that $y = x$, it follows that $u = x$.

III.2 *Skolem Functions and Kreisel Counter-examples*

We describe an alternative interpretation that will be useful later on. It consists in decomposing Eloise's strategies into *Skolem functions* and Abelard's strategies into *Kreisel counterexamples*.

When φ is an IF formula, the skolemized form or skolemization of φ with free variables in U, $Sk_U(\varphi)$, is defined by induction on the subformulas of φ:

1. $Sk_U(\psi) = \psi$, for ψ a literal
2. $Sk_U(\psi \circ \theta) = Sk_U(\psi) \circ Sk_U(\theta)$, for $\circ \in \{\vee, \wedge\}$
3. $Sk_U((\forall x/W)\psi) = \forall x Sk_{U \cup \{x\}}(\psi)$
4. $Sk_U((\exists x/W)\psi) = Sub(Sk_{U \cup \{x\}}(\psi), x, f(y_1, ..., y_n))$

where $y_1, ..., y_n$ are all the variables in U - W and f is a new function symbol of appropriate arity. We abbreviate $Sk\, \emptyset(\varphi)$ by $Sk(\varphi)$.

Truth in a model $\mathbb{M}$, with respect to the assignment s which includes the free variables of φ is then defined by:

• $\mathbb{M}$, $\vDash^{+}_{SK} \varphi$ if and only if there exist functions $g_1, ..., g_n$ of appropriate arity in the universe M of $\mathbb{M}$ to be the interpretations of the new function symbols in $Sk_U(\varphi)$ such that

$$\mathbb{M}, g_1, ..., g_n, s \vDash Sk_U(\varphi)$$

where U is the domain of s. The functions $g_1, ..., g_n$ are called skolem functions.

Recall one of our earlier examples

$$\forall x \forall z (x \neq z \vee (\exists y/\{x\}) x = y).$$

Its Skolemized form is

$$\forall x \forall z (x \neq z \vee x = f(z))$$

where f is a new function symbol. Note that the earlier informal idiom "∃ sees z" is now given a precise formulation in the strategy function f taking z as its argument. It is straightforward to see that for every model $\mathbb{M}$ there is an expansion $\mathbb{M}$, g (g is the interpretation in M of the new function symbol f) such that

$$\mathbb{M}, g \vDash \forall x \forall z(x \neq z \vee x = f(z)).$$

We let g be defined by: $g(a) = a$ for every $a \in M$. Applying the definition $\mathbb{M}, s \vDash^{+}_{SK} \varphi$ we conclude that

$$\mathbb{M}, s \vDash^{+}_{SK} \forall x \forall z(x \neq z \vee (\exists y/\{x\})x = y).$$

We now define the dual procedure of Skolemization. The Kreisel form $Kr_U(\varphi)$ of the IF formula φ in negation normal form with free variables in U is defined by:

1. $Kr_U(\psi) = \psi$, for ψ a literal
2. $Kr_U(\psi \vee \theta) = Kr_U(\psi) \wedge Kr_U(\theta)$
3. $Kr_U(\psi \wedge \theta) = Kr_U(\psi) \vee Kr_U(\theta)$
4. $Kr_U((\exists x/W)\psi) = \forall x Kr_{U \cup \{x\}}(\psi)$
5. $Kr_U((\forall x/W)\psi) = Sub(Kr_{U \cup \{x\}}(\psi), x, g(y_1, ..., y_m))$

where $y_1, ..., y_m$ are all the variables in $U - W$.

Falsity in a model $\mathbb{M}$ with respect to the assignment s which includes the free variables of φ is then defined by:

- $\mathbb{M}, s \vDash^{-}_{SK} \varphi$ if and only if there exist $h_1, ..., h_m$ in M to be the interpretations of the new function symbols in $Kr(\varphi)$ such that

  $$\mathbb{M}, h_1, ..., h_m, s \vDash Kr_U(\varphi)$$

  where U is the domain of s. We call $h_1, ..., h_m$ Kreisel counterexamples.

We illustrate it with the second of our earlier examples:

$$\exists x \exists y \exists z[x = y \wedge \forall v(\forall u/\{x\})(u \neq x \vee v \neq z)].$$

Its Kreisel form is

$$\forall x \forall y \forall z[x = y \vee (f(y, z, g(x, y, z)) = x \wedge v = z)]$$

where f and g are new function symbols. Let $\mathbb{M}$ be a model with at least two elements. In order to show that our initial sentence is false in $\mathbb{M}$, we show that we can find the Kreisel counter-examples h_1 (the interpretation of g) and h_2 (the interpretation of f) such that

$$\mathbb{M}, h_1, h_2 \vDash \forall x \forall y \forall z[x \neq y \vee (f(y, z, g(x, y, z)) = x \wedge g(x, y, z) = z)].$$

We let: $h_1(a, b, c) = c$ and $h_2(a, b, c) = b$.

IV. Indeterminacy

Imperfect information introduces indeterminacy into the games. We consider two examples:

$$\varphi_{MP} = \forall x(\exists y/\{x\})x = y$$

and

$$\varphi_{IMP} = \forall x(\exists y/\{x\})x \neq y.$$

The first one "expresses" the *Matching Pennies* game in IF logic: two players turn a coin to Head or Tail independently of each other. If their choices match, the second player wins. Otherwise the first player wins. We take the second sentence to express the *Inverted Matching Pennies*.

Using the Skolem-Kreisel interpretation mentioned earlier, we can show that both sentences are indeterminate on every structure which contains at least two elements. In order to see this, notice that the Skolem and Kreisel forms of φ_{MP} are $\forall xx = c$ and $\forall yd \neq y$, respectively, where c and d are new constants. Similarly, the Skolem and Kreisel forms of φ_{IMP} are $\forall yd \neq y$ and $\forall xx = c$, respectively.

The first claim is straightforward. Let $\mathbb{M}$ be a set which contains at least two elements. Obviously there is no extension $\mathbb{M}, a$ of $\mathbb{M}$ such that $\mathbb{M}, a \vDash \forall xx = c$. And by analogy, there is no extension $\mathbb{M}, b$ of $\mathbb{M}$ such that $\mathbb{M}, b \vDash \forall yd \neq y$. Thus by the definitions above, φ_{MP} is neither true nor false in $\mathbb{M}$. A similar argument establishes that φ_{IMP} is neither true nor false in $\mathbb{M}$.

V. Equilibrium Semantics

To resolve the indeterminacy of IF sentences, we move to strategic games. This technique, well known to game theorists, has been described for the first time in [Sevenster (2006)] developed in [Sevenster and Sandu (2010)], [Mann, Sandu, and Sevenster (2011)], [Sandu (2012)], [Sandu (forthcoming)], and [Barbero and Sandu (forthcoming)]. We first sketch the intuitive idea for the two sentences above and a model (which is a set) $\mathbb{M}=\{1, 2, 3, 4\}$. Given the equivalence of the three semantic interpretations listed earlier, we can take a strategy for Eloise in the semantical game $G(\mathbb{M}), \varphi_{MP})$ to be a sequence of functions to be the interpretations in $\mathbb{M}$ of the new function symbols in the Skolem form of φ_{MP}. In the case of φ_{MP} there is only one new function symbol: the constant c. Whence a strategy for Eloise is any individual $n \in \mathbb{M}$. By a similar argument, a strategy for Abelard reduces to any individual $m \in \mathbb{M}$. Now when Eloise plays n and Abelard plays m, a play of the semantical game $G(\mathbb{M}, \varphi_{MP})$ is generated. This play is a win for Eloise if $n = m$, and a win for Abelard if $n \neq m$.

We are now ready for the general definition of strategic IF games.

A strategic IF game Γ has the form $\Gamma(\mathbb{M},\varphi) = (S_\exists, S_\forall, u_\exists, u_\forall)$ where:

- $S_\exists$ is the set of strategies of Eloise in the semantical game $G(\mathbb{M},\varphi)$
- $S_\forall$ is the set of strategies of Abelard in the semantical game $G(\mathbb{M},\varphi)$
- $u_\exists$ the payoff function of Eloise: $u_\exists(s, t) =1$ if playing $s \in S_\exists$ against $t \in S_\forall$ yields a win for $\exists$ in $G(\mathbb{M},\varphi)$; and $u_\exists (s, t) = 0$, otherwise.
- $u_\forall$ is defined analoguously.

It is customary to present a strategic game in a matrix form. When φ is φ_{MP} and $M = \{1, 2, 3, 4\}$, the corresponding strategic IF game is represented by the matrix:

	1	2	3	4
1	(1,0)	(0,1)	(0,1)	(0,1)
2	(0,1)	(1,0)	(0,1)	(0,1)
3	(0,1)	(0,1)	(1,0)	(0,1)
4	(0,1)	(0,1)	(0,1)	(1,0)

Notice that strategic IF games are 2 player, win-loss games: For every $s \in S_\exists$ and $t \in S_\forall$ we have: $u_\exists(s, t) + u_\forall(s, t) = 1$.

The advantage of turning semantical games into strategic games is that we can now use solution concepts from classical strategic game theory. One such notion is that of two strategies being in *equilibrium*. For a strategic IF game $\Gamma(\mathbb{M},\varphi) = (S_\exists, S_\forall, u_\exists, u_\forall)$, the pair $(\sigma^*, \tau^*) \in S_\exists \times S_\forall$ is an equilibrium (in pure strategies) if the following two conditions are both satisfied:

1. $u_\exists (\sigma^*, \tau^*) \geq u_\exists (\sigma^*, \tau^*)$ for every $\sigma \in S_\exists$
2. $u_\forall (\sigma^*, \tau^*) \geq u_\forall (\sigma^*, \tau^*)$ for every $\tau \in S_\forall$.

It may be checked that in our two examples there is no equilibrium. This is, obviously, nothing else than the counterpart of the indeterminacy of sentence φ_{MP} on any model which contains at least two elements.

To resolve the indeterminacy of games, we follow a strategy well known to game theorists: we move to *mixed strategies*.

Let $\Gamma(\mathbb{M}, \varphi) = (S_\exists, S_\forall, u_\exists, u_\forall)$, be a finite strategic IF game. A mixed strategy ν for player p is a probability distribution over S_p, that is, a function $\nu: S_p \rightarrow [0, 1]$ such that $\Sigma_{\sigma \in Sp}\ \nu(\tau) = 1$. ν is uniform over $S'_i \subseteq S_i$ if it assigns equal probability to all strategies in S'_i and zero probability to all the strategies in S_i - S'_i.

Let $\Delta(S_p)$ be the set of mixed strategies over S_p. If $\mu \in \Delta(S_\exists)$ and $\nu \in \Delta(S_\forall)$, the *expected utility* for player p is given by:

$$U_p(\mu, \nu) = \Sigma_{\sigma \in S\exists}\Sigma_{\sigma \in S\forall}\mu(\sigma) \vee (\tau)\ u_p(\sigma,\tau).$$

We can identify a pure strategy $\sigma \in S_\exists$ with a "degenerate" mixed strategy which assigns to σ probability 1 and 0 to all the other strategies in $S_\exists$. That is, when $\sigma \in S_\exists$ and $\nu \in \Delta\ (S_\forall)$, we let

$$U_p(\sigma, \nu) = \Sigma_{\tau \in S\forall}\nu(\tau)u_p(\sigma,\tau).$$

Similarly, when $\tau \in S_\forall$ and $\mu \in \Delta(S_\exists)$, we let

$$U_p(\mu, \tau) = \Sigma_{\sigma \in S\exists}\mu(\sigma)\ u_p(\sigma,\tau).$$

Let $\Gamma = (S_\exists, S_\forall, u_\exists, u_\forall)$, be a two-player finite strategic game which is also a win-lose game (the only payoffs are 0 and 1). For $\mu^* \in \Delta(S_\exists)$ and $\nu^*\ (S_\forall)$, the definition of (μ^*, ν^*) being a mixed strategy equilibrium in Γ is completely analogue to the earlier one.

The following results are well known.

Theorem (von Neuman's Minimax Theorem) Every finite, two-person, win-lose game has an equilibrium in mixed strategies.

Corollary Let (μ,ν) and (μ',ν') be two mixed strategy equilibria in a win-lose game. Then $U_p(\mu,\nu) = U_p(\mu',\nu')$.

The above results tell us that for two-player finite win-lose games an equilibrium always exists (von Neumann's theorem), and in addition, any two mixed strategy equilibria deliver the same expected utility. We shall take *the value of the strategic game* to be the expected utility delivered by any of the mixed strategy equilibria in the game.

Mann et al (2011) develop a toolkit for identifying mixed strategy equilibria in strategic IF games. Here we give one such result.

Proposition Let $\Gamma(\mathbb{M}, \varphi) = (S_\exists, S_\forall, u_\exists, u_\forall)$ be a 2-player, finite strategic game. Let $\mu^* \in \Delta(S_\exists)$ and $\nu^* \in \Delta(S_\forall)$. The pair (μ^*,ν^*) is an equilibrium in Γ if and only if the following conditions hold:

1 $U_\exists(\mu^*, \nu^*) = U_\exists(\sigma, \nu^*)$ for every $\sigma \in S_\exists$ in the support of μ^*

2. $U_\forall(\mu^*,\nu^*) = U_\forall(\mu^*, \tau)$ for every $\tau \in S_\forall$ in the support of ν^*

3. $U_\exists(\mu^*, \nu^*) \geq U_\exists(\sigma, \nu^*)$ for every $\sigma \in S_\exists$ outside the support of μ^*

4. $U_\forall(\mu^*, \nu^*) \geq U_\forall(\mu^*,\tau)$ for every $\tau \in S_\forall$ outside the support of ν^*

(The support of a mixed strategy is the set of pure strategies which are assigned non-zero probability). We illustrate the notions introduced so far by returning to the two examples of indeterminate IF sentences considered so far. Recall the IF strategic game $\Gamma(\mathbb{M}, \varphi_{MP})$ when $\mathbb{M} = \{1, 2, 3, 4\}$:

	1	2	3	4
1	(1,0)	(0,1)	(0,1)	(0,1)
2	(0,1)	(1,0)	(0,1)	(0,1)
3	(0,1)	(0,1)	(1,0)	(0,1)
4	(0,1)	(0,1)	(0,1)	(1,0)

Let μ be a uniform strategy for $\exists$ over $\mathbb{M}$ and ν be a uniform strategy for $\forall$ over the same set. Eloise's expected utility for the pair (μ, ν) is

$U_\exists(\mu,\nu) = 4 \times (¼ \times ¼ \times 1) = ¼$

and Abelard's expected utility for the same pair is

$$U_{\forall}(\mu, \nu) = 4 \times (¼ \times ¼ \times 3) = ¾$$

It is straightforward to show, using the last Proposition, that the pair of mixed strategies (μ, ν) is an equilibrium. We conclude that the value of the game is $U_{\exists}(\mu,\nu)$ = ¼. In the general case in which $\mathbb{M}$ has n elements, it may be checked that the pair (μ, ν) of uniform strategies over $\mathbb{M}$ is an equilibrium in the game and the value of the game is $1/n$.

Let φ be an IF sentence and $\mathbb{M}$ a finite model. We shall take *the value of the sentence* φ in $\mathbb{M}$ to be the value of the strategic game $\Gamma(\mathbb{M}, \varphi)$. Thus the value of φ_{MP} in $\mathbb{M} = \{1, ..., n\}$ is $1/n$.

When $\mathbb{M} = \{1, 2, 3, 4\}$, the strategic IF game Γ ($\mathbb{M}$, φ_{MP}) is represented below:

	1	2	3	4
1	(0,1)	(1,0)	(1,0)	(1,0)
2	(1,0)	(0,1)	(1,0)	(1,0)
3	(1,0)	(1,0)	(0,1)	(1,0)
4	(1,0)	(1,0)	(1,0)	(0,1)

It may be checked that the pair (μ, ν) of uniform probability distributions over $\mathbb{M}$ is also an equilibrium in this case. Thus the value of φ_{MP} in $\mathbb{M}$ is ¾.

VI. Game-theoretical Probabilities

We shall internalize the probabilistic interpretation of IF logic by extending the object language to include identities of the form $NE(\varphi) = r$. Their interpretation is straightforward (the idea is due to Galliani and appears in Sandu, forthcoming):

$\mathbb{M} \vDash NE(\varphi) = r$ if and only if the value of φ in $\mathbb{M}$ is r.

It follows, from results in [Mann et al. (2011), chapter 7] that the probabilistic interpretation of IF logic is a conservative extension of the game-theoretical interpretation in the following sense:

(i) $\mathbb{M}\ s \vDash^{+}_{GTS} \psi$ iff $\mathbb{M} \vDash NE(\psi) = 1$

(ii) $\mathbb{M}\ s \vDash^{-}_{GTS} \psi$ iff $\mathbb{M} \vDash NE(\psi) = 0$

Mann et al. (Chapter 7) also show the following:

P1. $NE(\varphi \vee \psi) = \max(NE(\varphi), NE(\psi))$
P2. $NE(\varphi \wedge \psi) = \min(NE(\varphi), NE(\psi))$
P3. $NE(\neg\varphi) = 1 - NE(\varphi)$.

From these facts, it is easy to establish the validity of the following principles:

Ax1. $NE(\varphi) \geq 0$
Ax2. $NE(\varphi) + NE(\neg\varphi) = 1$
Ax3. $NE(\varphi) + NE(\psi) \geq NE(\varphi \vee \psi)$
Ax4. $NE(\varphi \wedge \psi) = 0 \rightarrow NE(\varphi) + NE(\psi) = NE(\varphi \vee \psi)$

(Ax1) follows automatically from the probabilistic interpretation. Ax2 is the counterpart of (P3). (Ax3) follows from (Ax1) and (P2). As for (Ax4), let $\mathbb{M}$ be an arbitrary model and suppose that $\mathbb{M} \vDash NE(\varphi \wedge \psi) = 0$. Then by (ii), $\mathbb{M}\ s \vDash^{-}_{GTS} (\varphi \wedge \psi)$. From the way strategies are defined in semantical games: $\mathbb{M}\ s \vDash^{-}_{GTS} \varphi$ or $\mathbb{M}\ s \vDash^{-}_{GTS} \psi$. Suppose that $\mathbb{M}\ s \vDash^{-}_{GTS} \varphi$. Then by (ii) $\mathbb{M} \vDash NE(\varphi) = 0$ and it also follows that $\mathbb{M} \vDash_{GTS} NE(\varphi) + NE(\psi) = NE(\psi)$. From P1 we also know that $\mathbb{M} \vDash_{GTS} NE(\varphi \vee \psi) = \max(NE(\varphi), NE(\psi))$, i.e., $\mathbb{M} \vDash_{GTS} NE(\varphi \vee \psi) = NE(\psi)$. The other case is similar.

VII. Two Conceptions of Probability: Statistical Knowledge vs. Degree of Belief

It is common knowledge that there are, roughly, two interpretations of probabilities:

- Probabilities as proportions or relative frequencies
- Probabilities as degrees of belief

Both of them obey the so-called Kolmogorov axioms (Here S is a sample space and Π is a field of subsets):

1. $pr(A) \geq 0$ for all $A \in \Pi$
2. $pr(S) = 1$
3. If $pr(A \cap B) = 0$ then $pr(A \cup B) = pr(A) + pr(B)$

Bacchus (1990) gives a nice overview of the two conceptions and illustrates them by the following statements:

1. The probability that a randomly chosen bird flies is greater than 0.9
2. The probability that Tweety (a particular bird) flies is greater than 0.9

(1) seems to assume one world (the real world) and in this world some probability distribution over the set of birds. It says that if we consider a bird chosen at random, it will fly with probability greater than 0.9. Halpern (1990) endorses the same distinction and takes (1) to be about a chance set up: the result of performing an experiment or trial in some situation. Given some statistical information (that 90% of the individuals in a population have property P), then we may imagine a chance set up in which a randomly chosen individual has probability 0.9 of having property P.

For both Bacchus and Halpern (2) is about the degree of belief of the agent and thus seems to implicitly assume a number of possibilities (possible worlds), in some of which Tweety flies, while in others it does not fly, and a probability distribution over them. Thus the contrast between statistical probability and degree of belief is spelled out in terms of two kinds of probability distributions: one over the individuals of the (one world) universe, and the other over the set of relevant possible worlds. Corresponding to them, two logical formalisms have been developed, both containing an empirical component in the semantics.

VII.1 *Statistical Probabilities*

Probability distributions of the first kind are represented in a logical formalism which extends first-order languages with expressions of the form $w_x(\varphi)$. Here $w_x(\varphi)$ is a term and w_x is a binding expression which binds the free occurrences of the variable x in the first-order formula φ. In this setting, sentences like (1) have the logical form $w_x(\varphi) > 0.9$. The syntax actually allows arbitrary sequences of distinct variables in the subscript. To fix intuitions, we give an example (taken from Halpern).

We let $S(x, y)$ say that x is the son of y. The three terms below have the following interpretations:

- $w_x(S(x, y))$: the probability that a randomly chosen x is the son of y
- $w_y(S(x, y))$: the probability that x is the son of a randomly chosen y
- $w_{(x,\, y)}(S(x, y))$: the probability that a randomly chosen pair (x, y) has the property that x is the son of y

The syntax is two sorted, with the variables *x, y...* being object variables. I will skip over the details.

Formulas of the object language are interpreted on probability models. These are tuples $M = (D, I, \mu)$ where (D, I) is an ordinary first-order model and μ is a discrete probability function on D. That is, μ is a mapping from D to the real interval [0, 1] such that $\Sigma_{d \in A}\mu(d) = 1$. For any $A \subseteq D$ we define:

$$\mu(A) = \Sigma_{d \in A}\mu(d)$$

Given such a probability function we can define a discrete probability function μ^n on D^n by taking

$$\mu^n(d_1, ..., d_n) = \mu(\mathrm{d}_1) \times ... \times \mu(d_n).$$

The evaluation of formulas in probability structures follows the standard lines. The clause which interests us is (here s is an assignment)

- $[w_{(x1, ..., xn)}(\varphi)]_{M,s} = \mu^n(\{(d_1, ..., d_n) : (M,s[x_1/d_1, ..., x_n/d_n]) \vDash \varphi\})$.

We return to the last example and consider a model $M = (\{a, b, c\}, I, \mu)$ such that $I(S) = \{(a, b)\}$ and $\mu(a) = 1/3$, $\mu(b) = ½$ and $\mu(c) = 1/6$. Let s be an assignment such that $s(x) = a$ and $s(y) = c$. Then

$$[w_x(Son(x, y)]_{(M, s)} = 0$$
$$[w_y(Son(x, y)]_{(M, s)} = ½$$
$$[w_{x, y}(Son(x, y)](_{M, vs}) = 1/6$$

VII.2 *Degrees of Belief*

Corresponding to probability distributions over possible worlds, we have extensions of propositional or first-order languages with terms of the form $t(\varphi)$.The intended interpretation is: the degree of belief that φ. In this new setting sentences like (2) are represented by

$$t(Flies(Tweety)) > 0.9.$$

Let us mention right away the syntactical distinction between $w_x(\varphi)$ and $t(\varphi)$: in the former an occurrence of a free variable x in φ is bound by w_x whereas in the later t is not a binding expression. That will be seen to have important consequences (cf. the Lemma below).

The formulas of a given object language are interpreted on probability models which now have the form (D, W, π, μ): D is a domain, W is a set of possible worlds, and for each $w \in W$, $\pi(w)$ assigns to the predicate and function symbols of the language predicates and functions of the right arity over D. μ is a discrete probability function on W. Here are few clauses which interest us:

- $(M, w, s) \vDash P(x)$ iff $s(x) \in \pi(w)(P)$
- $(M, w, s) \vDash t_1 = t_2$ iff $[t_1]_{(M, w, s)} = [t_2]_{(M, w, s)}$
- $(M, w, s) \vDash \forall x\varphi$ iff $(M, w, s(x/d)) \vDash \varphi$ for each $d \in D$
- $[t(\varphi)]_{(M, w, s)} = \mu(\{w' \in W : (M, w, s) \vDash \varphi\})$

That is, the predicate Flies receives an extension in every possible world. As a result *Flies*(*Tweety*) is true in some possible worlds but not in others. A probability distribution μ is then assigned to the set of possible worlds. Finally we check whether the set of possible worlds where *Flies*(*Tweety*) has probability greater that 0.9.

We note that when the probabilities are all 0 and 1, this variant of the probability logic reduces to ordinary logic.

The following Lemma (proved in [Halpern (1990)]) shows the main difference between the two approaches:

Lemma If φ is a closed formula, then for any vector $\vec{x}$ of distinct object variables we have

$$\vDash \mathrm{W}_{\vec{x}}(\varphi) = 0 \vee \mathrm{W}_{\vec{x}}(\varphi) = 1$$

This Lemma holds for the first approach and not for the second. It tells us that closed sentences behave classically in the first approach. It follows that a sentence like *Flies*(*Tweety*) cannot take intermediate values between 0 and 1, if represented in the first approach. It is thus inconsistent to claim that the probability of *Flies*(*Tweety*) is, say, less than 0.95 and greater than 0.9 in the first approach. But this is not any longer so in the current approach where *Flies*(*Tweety*) is prefixed with the operator *t*.

VIII. Game-theoretical Probabilities Reconsidered

We have analyzed three logical frameworks which correspond to three probability distributions:

- $w_{\vec{x}}(\varphi)$ arises out of probability distributions over the individuals in a single universe

- $t(\varphi)$ arises out of probability distributions over possible worlds
- $NE(\varphi)$ arises out of the equilibria of probability distributions over strategies (functions) over a single universe

All of them obey Kolmogorov probability axioms. Syntactically $NE(\varphi)$ operates in the same way as $t(\varphi)$: neither NE nor t has a binding role. This is an interesting fact in itself given that there is a general agreement in the game-theoretical literature that the notion of (Nash) equilibrium cannot be properly understood without reference to the players' beliefs. The connection between Nash equilibria and belief deserves much more space than we have here. We prefer instead to work out an example which illustrates the reduction of game-theoretical probabilities to probability distributions of the first kind.

Recall our earlier example

$$\varphi_{MP} = \forall x(\exists y/\{x\})x = y$$

and the model $\mathbb{M} = \{1, 2, 3, 4\}$.

Recall also that $Sk(\varphi_{MP}) = \forall xx = c$, $Kr(\varphi_{MP}) = \forall yd \neq y$ and the strategic game Γ ($\mathbb{M}$, φ_{MP}) has the matrix form:

	1	2	3	4
1	(1,0)	(0,1)	(0,1)	(0,1)
2	((0,1)	(1,0)	(0,1)	(0,1)
3	(0,1)	(0,1)	(1,0)	(0,1)
4	(0,1)	(0,1)	(0,1)	(1,0)

We pointed out that the pair (μ, ν) of two uniform probability distributions over $\mathbb{M}$ constitute an equilibrium in the game.

We now convert $Sk(\varphi_{MP})$ into the "statistical" sentence

$$w_x(x = c)$$

Next we enlarge the model $\mathbb{M}$ to a probability model in two steps:

1. We expand $\mathbb{M}$ to the model ($\mathbb{M}$, a) where a is the interpretation of the new constant symbol c: given the uniform probability distribution μ, we take a to be any of the four elements of $\mathbb{M}$, say $a = 1$.
2. We expand the model ($\mathbb{M}$, a) to the probability model $\mathbb{M}^* = (\mathbb{M}, 1, \nu)$.

We can check that in the Halpern-Bacchus semantics we have:

$$[w_x(x = c)]\mathbb{M}^* =$$

$$\nu(\{a : (\mathbb{M}^*, a) \vDash x = c\}) =$$

$$\nu(\{1\}) = ¼$$

We have "reduced" the value of the game given by the expected utility returned to Eloise by the mixed strategy equilibrium (μ, ν) to a statistical, relative frequency value, as modeled in the Bacchus-Halpern approach. Notice that

$$\nu(\{1\}) = \Sigma_{n=1, \dots, 4}\, \nu(n)$$

$$= U_\exists(1, \nu)$$

$$= U_\exists(\mu, \nu)$$

holds in virtue of the last proposition of section V.

This example has a philosophical significance: it shows how in some simple cases, a statistical value ($[w_x(x = c)]\mathbb{M}^*$) which is based on an empirical component in the semantics is obtained from a non-empirical, mathematical concept: the value $U_\exists(\mu, \nu)$ of the game.

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